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Geodesic connectedness of stationary spacetimes with optimal growth

R. Bartolo[a], A.M. Candela[b,*], J.L. Flores[c]

[a] *Dipartimento di Matematica, Politecnico di Bari, Via G. Amendola 126/B, 70126 Bari, Italy*
[b] *Dipartimento di Matematica, Università degli Studi di Bari, Via E. Orabona 4, 70125 Bari, Italy*
[c] *Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias, Universidad de Málaga, Campus Teatinos, 29071 Málaga, Spain*

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Abstract

In the last fifteen years variational methods have been widely applied in the study of geodesic connectedness of stationary spacetimes. In this paper we introduce fine estimates which allow us to apply such methods to this problem in an optimal way, improving by far previous results on the subject. Our estimates also seem useful for extending the existing results in other related subjects, for example, connectedness by timelike geodesics and existence of normal trajectories.

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1. Introduction

A Lorentzian manifold is called *stationary* if it admits a timelike Killing vector field. An important sub-class is characterized by the following simpler structure:

Definition 1.1. A Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ given by a global splitting $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ is *(standard) stationary* if $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ is a finite dimensional connected Riemannian manifold and

* Corresponding author. Tel.: +39 080 5442711; fax: +39 080 5963612.
E-mail address: candela@dm.uniba.it (A.M. Candela).

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the metric is

$$\langle \zeta, \zeta' \rangle_L = \langle \xi, \xi' \rangle + \langle \delta(x), \xi \rangle \tau' + \langle \delta(x), \xi' \rangle \tau - \beta(x) \tau \tau' \tag{1.1}$$

for any $z = (x, t) \in \mathcal{M}$ and $\zeta = (\xi, \tau)$, $\zeta' = (\xi', \tau') \in T_z\mathcal{M} = T_x\mathcal{M}_0 \times \mathbb{R}$, where δ and β are respectively a smooth vector field and a smooth strictly positive scalar field on $\mathcal{M}_0$.

In the particular case of $\delta \equiv 0$ the Lorentzian manifold is called *(standard) static*.

Stationary spacetimes play an important role in General Relativity, since they represent time-independent gravitational fields which may arise as final states of multiple astrophysical processes (collapsing stars,...). The most well-known example of purely stationary spacetime is the classical Kerr spacetime, which essentially represents the gravitational field outside a rotating star (more details and examples can be found in [4,11,14]).

Recall that by a geodesic z in a Lorentzian (semi-Riemannian) manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ we mean a smooth curve $z : I \to \mathcal{M}$ (I a real interval) such that

$$D_s \dot{z}(s) = 0 \qquad \text{for all } s \in I,$$

where D_s denotes the covariant derivative along z induced by the Levi-Civita connection of metric $\langle \cdot, \cdot \rangle_L$. It is well known that, if z is a geodesic, then

$$E_z \equiv \langle \dot{z}(s), \dot{z}(s) \rangle_L \quad \text{for all } s \in I$$

is a constant. Thus, in a Lorentzian manifold geodesics can be classified according to the sign of E_z (causal character): a geodesic z is said to be timelike (resp. lightlike; causal; spacelike) if E_z is negative (resp. null; non-positive; positive).

Apart from their purely geometric interest, geodesics are very relevant in General Relativity, since causal ones represent trajectories of particles or light rays under the action of the gravitational field. Therefore, the study of their properties is of interest from both the mathematical and the physical viewpoint.

In the last few years, there has been intensive research devoted to the study of different properties of geodesics in stationary spacetimes, such as geodesic connectedness and completeness, existence of periodic trajectories, multiplicity and causal character of connecting geodesics. Remarkably, in the static case some of these properties present a "critical" behavior with respect to a quadratic asymptotic growth of metric coefficient β[1] (a nice survey on this subject can be found in [16]). In fact, in [3] the authors studied the geodesic connectedness of static spacetimes, that is, they asked whether any two points of the spacetime can be connected by a geodesic. They answered positively to this question whenever coefficient β grows (at most) quadratically [3, Theorem 1.1]. Moreover, this growth is optimal/critical, in the sense that static counterexamples are found with superquadratic β presenting a growth as close as we want to the quadratic one [3, Section 7]. Even more relevant, this analysis is totally fulfilled by using variational techniques, showing that the variational approach is specially well-adapted to this problem.

Taking into account these considerations, the following questions arise in a natural way: Does geodesic connectedness also present a critical behavior in the more general class of stationary spacetimes? Can variational methods provide this extended optimal result on their own?

In [15, Corollary 3.4] Sánchez proved global hyperbolicity of stationary spacetimes by assuming a quadratic growth for coefficient β and a linear one for δ; conditions which are also

[1] In the particular case of geodesic completeness, the critical behavior is with respect to a quadratic asymptotic growth of the reciprocal β^{-1}.

optimal. Taking into account that global hyperbolicity implies causal geodesic connectedness between causally related points, this result suggests an affirmative answer to the first question, providing also the right growths to impose on β and δ. The aim of this paper is to exploit variational tools by using fine estimates in order to ensure that these conditions on β and δ are indeed sufficient for geodesic connectedness. In addition, we will prove that our result essentially exhausts the variational technique, suggesting its optimal/critical character.

The history of the problem of geodesic connectedness in stationary spacetimes is long. The first result on the subject appeared in [9, Theorem 1.10] (see also [12, Theorem 3.4.3]). There, Giannoni and Masiello proved geodesic connectedness when β is far away from 0 and bounded from above, i.e., $\nu, k_1 > 0$ exist such that

$$\nu \leq \beta(x) \leq k_1 \qquad \text{for all } x \in \mathcal{M}_0, \tag{1.2}$$

and δ is bounded, i.e., $\sup_{x\in\mathcal{M}_0}\langle\delta(x), \delta(x)\rangle < +\infty$. In a subsequent paper Pisani relaxed previous hypotheses by only assuming a sublinear growth for β and δ, i.e., there exist $\nu > 0$, $\mu_1, \mu_2 \geq 0$, $k_1, k_2 \in \mathbb{R}$, $q_1, q_2 \in [0, 1[$ and a point $\bar{x} \in \mathcal{M}_0$ such that

$$\nu \leq \beta(x) \leq \mu_1 d^{q_1}(x, \bar{x}) + k_1 \qquad \text{for all } x \in \mathcal{M}_0$$

and

$$\sqrt{\langle\delta(x), \delta(x)\rangle} \leq \mu_2 d^{q_2}(x, \bar{x}) + k_2 \qquad \text{for all } x \in \mathcal{M}_0 \tag{1.3}$$

where $d(\cdot, \cdot)$ is the distance induced on $\mathcal{M}_0$ by its Riemannian metric $\langle\cdot, \cdot\rangle$ (see [13, Theorem 1.2]). Several years later, in [10] Giannoni and Piccione also studied the problem in general (not necessarily standard) stationary Lorentzian manifolds. In particular, when restricted to the standard case, their result provides geodesic connectedness when β satisfies (1.2) and δ grows as in (1.3) (see [10, Appendix A]). Finally, under the same hypotheses of [13], in [8] the authors provided existence and also multiplicity results for geodesics connecting two submanifolds (in particular, two points) in a stationary spacetime.

In this paper we improve by far the hypotheses needed to ensure geodesic connectedness for a stationary spacetime: for the positive coefficient β only a *quadratic* asymptotic growth is needed while any other lower bound is ruled out (in fact, in all the above cited papers the lower bound in (1.2) was required unlike in the static case); additionally, for δ we assume a *linear* growth. As in most of the papers on the topic, we will apply variational methods. However, original accurate estimates are introduced here. As a consequence, our approach exhausts the variational technique, suggesting that our result is optimal/critical (see Example 2.7 and its previous discussion).

Our main theorem can be stated as follows.

Theorem 1.2. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a (standard) stationary Lorentzian manifold as in Definition 1.1. Suppose that*

(H_1) *Riemannian manifold $(\mathcal{M}_0, \langle\cdot, \cdot\rangle)$ is complete and smooth (at least C^3);*
(H_2) *there exist $\lambda, \mu \geq 0$, $k_1, k_2 \in \mathbb{R}$ and a point $\bar{x} \in \mathcal{M}_0$ such that*

$$\beta(x) \leq \lambda d^2(x, \bar{x}) + k_1 \qquad \text{for all } x \in \mathcal{M}_0, \tag{1.4}$$

$$\sqrt{\langle\delta(x), \delta(x)\rangle} \leq \mu d(x, \bar{x}) + k_2 \qquad \text{for all } x \in \mathcal{M}_0. \tag{1.5}$$

Then, $\mathcal{M}$ is geodesically connected.

Remark 1.3. If, in addition to the assumptions of Theorem 1.2, $\mathcal{M}_0$ is non-contractible in itself, then Ljusternik–Schnirelman Theory directly implies that any two points in $\mathcal{M}$ are joined by a sequence $(z_k)_k$ of spacelike geodesics such that $E_{z_k} \equiv \langle \dot{z}_k, \dot{z}_k \rangle_L \to +\infty$ (see, e.g., [3, Theorem 1.1]).

We finish this section by highlighting that the arguments firstly introduced in this paper can be also applied to other related questions. For example, the same ideas provide existence and multiplicity results for timelike geodesics connecting two points (see [1]),[2] and existence of connecting trajectories under the action of an external field in a stationary spacetime (see [2]).

2. Proof of Theorem 1.2

Let $\mathcal{M}$ be a (standard) stationary Lorentzian manifold and fix $z_0 = (x_0, t_0)$, $z_1 = (x_1, t_1)$ in $\mathcal{M}$. In what follows, we can choose $I = [0, 1]$ since the set of geodesics in a semi-Riemannian manifold is invariant by affine reparametrizations. Obviously, without restriction, in assumption (H_2) we can take $\bar{x} = x_0$ and replace (1.4) with

$$\beta(x) \le \lambda d^2(x, x_0) + 1 \qquad \text{for all } x \in \mathcal{M}_0 \tag{2.1}$$

and (1.5) with

$$\sqrt{\langle \delta(x), \delta(x) \rangle} \le \sqrt{\lambda} d(x, x_0) + 1 \qquad \text{for all } x \in \mathcal{M}_0. \tag{2.2}$$

By the product structure of $\mathcal{M}$, the infinite dimensional manifold $H^1(I, \mathcal{M})$ (first Sobolev space of curves on $\mathcal{M}$) is diffeomorphic to the product manifold $H^1(I, \mathcal{M}_0) \times H^1(I, \mathbb{R})$ and can be equipped with the structure of a Riemannian manifold by setting

$$\langle \zeta, \zeta \rangle_1 = \int_0^1 \langle \xi, \xi \rangle \mathrm{d}s + \int_0^1 \langle D_s \xi, D_s \xi \rangle \mathrm{d}s + \int_0^1 \tau^2 \mathrm{d}s + \int_0^1 \dot{\tau}^2 \mathrm{d}s,$$

for any $z = (x, t) \in H^1(I, \mathcal{M})$ and $\zeta = (\xi, \tau) \in T_z H^1(I, \mathcal{M}) \equiv T_x H^1(I, \mathcal{M}_0) \times H^1(I, \mathbb{R})$ (here, D_s denotes the covariant derivative along x induced by the Levi-Civita connection of metric $\langle \cdot, \cdot \rangle$).

By the Nash Embedding Theorem, as $\mathcal{M}_0$ is at least C^3 we can assume that it is a submanifold of the Euclidean space $\mathbb{R}^N$, $\langle \cdot, \cdot \rangle$ is the restriction to $\mathcal{M}_0$ of the Euclidean metric on $\mathbb{R}^N$ and $d(\cdot, \cdot)$ is the corresponding distance, i.e.,

$$d(\bar{x}_1, \bar{x}_2) = \inf \left\{ \int_a^b |\dot{\gamma}| \mathrm{d}s : \gamma \in A_{\bar{x}_1, \bar{x}_2} \right\} = \inf \left\{ \mathrm{length}(\gamma) : \gamma \in A_{\bar{x}_1, \bar{x}_2} \right\}$$

where $\bar{x}_1, \bar{x}_2 \in \mathcal{M}_0$, $|\dot{\gamma}(s)| = \sqrt{\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle}$ and $\gamma \in A_{\bar{x}_1, \bar{x}_2}$ if $\gamma : [a, b] \to \mathcal{M}_0$ is a piecewise smooth curve joining $\bar{x}_1$ to $\bar{x}_2$.

Hence, it can be proved that $H^1(I, \mathcal{M}_0)$ can be identified with the set of absolutely continuous curves $x : I \to \mathbb{R}^N$ with square summable derivative such that $x(I) \subset \mathcal{M}_0$. Furthermore, as $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ is a complete Riemannian manifold, $H^1(I, \mathcal{M})$ is also a complete Riemannian manifold with respect to $\langle \cdot, \cdot \rangle_1$.

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[2] Although in this paper the main result [1, Theorem 1.3] was stated by assuming a strictly positive lower bound for β and a sublinear growth for δ, the argument given there actually proves the result just under the same hypotheses as those of Theorem 1.2 here.

Let Z be the smooth manifold of all $H^1(I,\mathcal{M})$-curves joining z_0 to z_1 and let $\Omega^1(x_0,x_1)$ denote the smooth submanifold of $H^1(I,\mathcal{M}_0)$ formed by all $H^1(I,\mathcal{M}_0)$-curves joining x_0 to x_1 in $\mathcal{M}_0$. Since $H^1(I,\mathcal{M})$ is diffeomorphic to a product manifold it follows that

$$Z \equiv \Omega^1(x_0,x_1) \times W(t_0,t_1),$$

where

$$W(t_0,t_1) = \{t \in H^1(I,\mathbb{R}) : t(0)=t_0,\, t(1)=t_1\} = H_0^1 + T^*$$

with

$$H_0^1 = \{t \in H^1(I,\mathbb{R}) : t(0) = 0 = t(1)\}$$

and

$$T^* : s \in I \mapsto t_0 + s\Delta_t \in \mathbb{R}, \qquad \Delta_t = t_1 - t_0.$$

Hence, $W(t_0,t_1)$ is a closed affine submanifold of $H^1(I,\mathbb{R})$ with tangent space $T_tW = H_0^1$ for all $t \in W(t_0,t_1)$. Moreover, for all $x \in \Omega^1(x_0,x_1)$ we have

$$T_x\Omega^1(x_0,x_1) = \{\xi \in T_xH^1(I,\mathcal{M}_0) : \xi(0) = 0 = \xi(1)\}.$$

Thus, taking any curve $z=(x,t) \in Z$ we have

$$T_zZ \equiv T_x\Omega^1(x_0,x_1) \times H_0^1,$$

and Z can be equipped with the following equivalent Riemannian structure:

$$\langle \zeta,\zeta\rangle_H = \langle (\xi,\tau),(\xi,\tau)\rangle_H = \int_0^1 \langle D_s\xi, D_s\xi\rangle \mathrm{d}s + \int_0^1 \dot{\tau}^2 \mathrm{d}s$$

for any $z=(x,t)\in Z$ and $\zeta = (\xi,\tau) \in T_zZ$.

According to Definition 1.1 action functional $f : Z \to \mathbb{R}$ is defined as

$$f(z) = \int_0^1 \langle \dot{z},\dot{z}\rangle_L \mathrm{d}s = \int_0^1 \left(\langle \dot{x},\dot{x}\rangle + 2\langle \delta(x),\dot{x}\rangle\dot{t} - \beta(x)\dot{t}^2\right)\mathrm{d}s$$

for any $z=(x,t)\in Z$. It is easy to prove that f is C^1 in Z with

$$\begin{aligned} f'(z)[(\xi,\tau)] = {} & 2\int_0^1 \langle \dot{x},\dot{\xi}\rangle \mathrm{d}s \\ & + 2\int_0^1 (\langle \delta'(x)[\xi],\dot{x}\rangle\dot{t} + \langle \delta(x),\dot{\xi}\rangle\dot{t} + \langle\delta(x),\dot{x}\rangle\dot{\tau})\mathrm{d}s \\ & - \int_0^1 \left(\beta'(x)[\xi]\dot{t}^2 + 2\beta(x)\dot{t}\dot{\tau}\right)\mathrm{d}s \end{aligned}$$

for any $z=(x,t)\in Z$, $(\xi,\tau)\in T_zZ$.

Standard "bootstrap" arguments show that $z^* = z^*(s)$ is a geodesic joining the two fixed points z_0 and z_1 in $\mathcal{M}$ (i.e., z_0 and z_1 are geodesically connected) if and only if z^* is a critical point of action functional f in Z.

But, unlike the Riemannian energy functional, the Lorentzian one f is unbounded both from above and from below and its critical points have infinite Morse index. Thus, in general, the existence of its critical levels cannot be directly investigated by means of classical topological methods.

Nevertheless, as the coefficients of the metric defined in (1.1) do not depend on the time variable t, the search for geodesics joining z_0 to z_1 can be reduced to the search for critical points of a functional depending only on the spatial component x (see the pioneer paper [6]).

To this end, let us consider functional $J : \Omega^1(x_0, x_1) \to \mathbb{R}$ defined as

$$J(x) = \int_0^1 \langle \dot{x}, \dot{x} \rangle \mathrm{d}s + \int_0^1 \frac{\langle \delta(x), \dot{x} \rangle^2}{\beta(x)} \mathrm{d}s - K_t^2(x) \int_0^1 \frac{1}{\beta(x)} \mathrm{d}s, \tag{2.3}$$

where

$$K_t(x) = \left(\Delta_t - \int_0^1 \frac{\langle \delta(x), \dot{x} \rangle}{\beta(x)} \mathrm{d}s \right) \left(\int_0^1 \frac{1}{\beta(x)} \mathrm{d}s \right)^{-1}. \tag{2.4}$$

The following variational principle can be stated (for more details, see [9, Theorem 2.2] or also [12, Theorem 3.3.2]).

Proposition 2.1. *Let $z^* = (x^*, t^*) \in Z$. The following statements are equivalent:*

(i) *z^* is a critical point of action functional f in Z;*

(ii) *x^* is a critical point of functional $J : \Omega^1(x_0, x_1) \to \mathbb{R}$ defined in (2.3) and $t^* = \Psi(x^*)$, with $\Psi : \Omega^1(x_0, x_1) \to W(t_0, t_1)$ such that*

$$\Psi(x)(s) = t_0 + \int_0^s \frac{\langle \delta(x(\sigma)), \dot{x}(\sigma) \rangle}{\beta(x(\sigma))} \mathrm{d}\sigma + K_t(x) \int_0^s \frac{1}{\beta(x(\sigma))} \mathrm{d}\sigma$$

and $K_t(x)$ defined as in (2.4).

Moreover, if (i)*, or equivalently* (ii)*, holds then $f(z^*) = J(x^*)$.*

Remark 2.2. From the proof of Proposition 2.1 as developed in [12, Section 3.3], it follows that taking any $x \in \Omega^1(x_0, x_1)$ we have

$$f'(x, \Psi(x))[(\xi, \tau)] = J'(x)[\xi] \quad \text{for all } \xi \in T_x\Omega^1(x_0, x_1), \tau \in H_0^1.$$

To prove the geodesic connectedness of $\mathcal{M}$ we shall apply the following classical theorem to functional J on the Riemannian manifold $\Omega^1(x_0, x_1)$.

Theorem 2.3. *Let Ω be a complete Riemannian manifold and J a C^1 functional on Ω bounded from below which satisfies the Palais–Smale condition, i.e., any $(x_k)_k \subset \Omega$ such that*

$$(J(x_k))_k \text{ is bounded and } \quad \lim_{k \to +\infty} J'(x_k) = 0$$

converges in Ω, up to subsequences. Then, J has a minimum point.

In order to ensure that the hypotheses of Theorem 2.3 hold under assumptions (2.1) and (2.2), previously we need some accurate estimates. They are included in the following three technical lemmas.

Lemma 2.4. *Let β satisfy (2.1). If $(x_k)_k \subset \Omega^1(x_0, x_1)$ satisfies $\|\dot{x}_k\| \to +\infty$ then*

$$\int_0^1 \frac{\|\dot{x}_k\|^2}{\beta(x_k)} \mathrm{d}s \to +\infty \qquad \text{as } k \to +\infty, \tag{2.5}$$

where $\|\dot{x}_k\|^2 = \int_0^1 \langle \dot{x}_k, \dot{x}_k \rangle \mathrm{d}s$.

Proof. The key arguments for deducing (2.5) are implicitly contained in the proof of [3, Proposition 4.1]. Nevertheless, for the sake of completeness, here we point out the main ideas.

Firstly, let us remark that the proof of (2.5) can be reduced to solving a 1-dimensional problem. Indeed, as in [7, Lemma 3.4], for any $x \in \Omega^1(x_0, x_1)$ we can define a new function $y : I \to \mathbb{R}$ such that

$$y(s) = \begin{cases} \displaystyle\int_0^s |\dot{x}(\sigma)|\mathrm{d}\sigma & \text{if } s \in [0, s_0] \\ \displaystyle\mathrm{d}(x_0, x_1) + \int_s^1 |\dot{x}(\sigma)|\mathrm{d}\sigma & \text{if } s \in]s_0, 1] \end{cases}$$

for a suitable $s_0 \in I$ such that y is continuous. As a consequence, we have

$$y \in W(0, d(x_0, x_1)) = \{y \in H^1(I, \mathbb{R}) : y(0) = 0, y(1) = d(x_0, x_1)\}.$$

By definition, it is

$$\int_0^1 |\dot{y}|^2 \mathrm{d}s = \|\dot{x}\|^2 \quad \text{and} \quad d(x(s), x_0) \le y(s) \quad \text{for all } s \in I. \tag{2.6}$$

Now, from (2.1) and (2.6), our task is reduced to proving that taking any $(y_k)_k$ in $W(0, d(x_0, x_1))$ such that

$$\alpha_k := \int_0^1 |\dot{y}_k|^2 \mathrm{d}s \to +\infty \qquad \text{if } k \to +\infty \tag{2.7}$$

then

$$\int_0^1 \frac{\alpha_k}{\lambda y_k^2 + 1} \mathrm{d}s \to +\infty \qquad \text{if } k \to +\infty. \tag{2.8}$$

Arguing by contradiction, if (2.8) does not hold then

$$\left(\int_0^1 \frac{\alpha_k}{\lambda y_k^2 + 1} \mathrm{d}s\right)_k \quad \text{has to be bounded} \tag{2.9}$$

(up to take a subsequence). Without loss of generality, by (2.7) we can assume $\alpha_k > 0$ and the existence of a sequence $(s_k)_k \subset I$ such that

$$y_k(s_k) \to +\infty \qquad \text{if } k \to +\infty \tag{2.10}$$

(otherwise $(\|y_k\|_\infty)_k$ has a bounded subsequence for which (2.8) holds in contradiction with (2.9)).

For simplicity, for all $k \in \mathbb{N}$ define the new functions

$$h_k(s) = \frac{|\dot{y}_k(s)|}{\sqrt{\alpha_k}} \quad \text{and} \quad g_k(s) = \sqrt{\frac{\alpha_k}{\lambda y_k^2(s) + 1}} \qquad \text{if } s \in I.$$

Clearly, the definition of α_k implies $\int_0^1 h_k^2 \mathrm{d}s = 1$. So, by the Cauchy–Schwarz inequality we have

$$\int_0^1 h_k g_k \mathrm{d}s \le \left(\int_0^1 \frac{\alpha_k}{\lambda y_k^2 + 1} \mathrm{d}s\right)^{1/2}, \tag{2.11}$$

where

$$\int_0^1 h_k g_k \mathrm{d}s = \int_0^1 \frac{|\dot{y}_k|}{\sqrt{\lambda y_k^2 + 1}} \mathrm{d}s = \frac{1}{\sqrt{\lambda}} \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}s} \operatorname{arcsinh}(\sqrt{\lambda}\ y_k) \right| \mathrm{d}s. \tag{2.12}$$

But, taking into account (2.10) we obtain

$$\begin{aligned} \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}s} \operatorname{arcsinh}(\sqrt{\lambda} y_k) \right| \mathrm{d}s &\geq \left| \int_0^{s_k} \frac{\mathrm{d}}{\mathrm{d}s} \operatorname{arcsinh}(\sqrt{\lambda} y_k) \mathrm{d}s \right| \\ &= |\operatorname{arcsinh}(\sqrt{\lambda} y_k(s_k))| \longrightarrow +\infty \quad \text{if } k \to +\infty. \end{aligned} \tag{2.13}$$

Hence, (2.11)–(2.13) give (2.8) in contradiction with (2.9). □

It is well known that when the Cauchy–Schwarz inequality becomes an equality for a function f, it must be a constant. The next lemma states that, even when the equality does not exactly hold, a very useful information about f can be deduced.

Lemma 2.5. *Let $(f_k)_k$ be a sequence of non-negative functions, $0 \not\equiv f_k \in L^2([0, a_k])$, and let a $c > 0$ exist such that*

$$c + \int_0^{a_k} f_k(r)\mathrm{d}r > \left(\int_0^{a_k} f_k^2(r)\mathrm{d}r \right)^{1/2} \sqrt{a_k} \qquad \text{for all } k \in \mathbb{N}. \tag{2.14}$$

Define

$$m_k := \frac{1}{a_k} \int_0^{a_k} f_k(r)\mathrm{d}r \qquad \text{and} \qquad D_k := \left\{ r \in [0, a_k] : f_k(r) \geq \frac{m_k}{2} \right\}.$$

If the measure of D_k, $|D_k|$, is written as

$$|D_k| = \epsilon_k a_k,$$

then we obtain

$$\epsilon_k > 1 - \frac{8c}{m_k a_k} - \frac{4c^2}{m_k^2 a_k^2} \qquad \text{for all } k \in \mathbb{N}.$$

Proof. Fix $k \in \mathbb{N}$. The proof is trivial if $\epsilon_k = 1$, i.e., $|D_k| = a_k$. On the other hand, it cannot be $\epsilon_k = 0$. In fact, if this happens then $|D_k^c| = a_k$ with

$$D_k^c := [0, a_k] \setminus D_k = \left\{ r \in [0, a_k] : f_k(r) < \frac{m_k}{2} \right\}.$$

Hence,

$$m_k a_k = \int_0^{a_k} f_k(r)\mathrm{d}r = \int_{D_k^c} f_k(r)\mathrm{d}r < \frac{m_k a_k}{2}$$

which is a contradiction.

So, assume $0 < \epsilon_k < 1$ and fix the following notation:

$$v_k := \frac{1}{\epsilon_k a_k} \int_{D_k} f_k(r)\mathrm{d}r, \qquad w_k := \frac{1}{(1 - \epsilon_k) a_k} \int_{D_k^c} f_k(r)\mathrm{d}r.$$

Clearly, by definition we have

$$m_k = v_k \epsilon_k + w_k (1 - \epsilon_k), \qquad 2w_k < m_k \leq 2v_k; \tag{2.15}$$

furthermore, by (2.15) it follows that $w_k < v_k$, and replacing this inequality into the equality of (2.15) we also obtain

$$m_k < v_k. \tag{2.16}$$

On the other hand, the Cauchy–Schwarz inequality implies

$$\int_{D_k} f_k^2(r)\mathrm{d}r \geq v_k^2 \epsilon_k a_k \qquad \text{and} \qquad \int_{D_k^c} f_k^2(r)\mathrm{d}r \geq w_k^2(1-\epsilon_k)a_k. \tag{2.17}$$

Therefore, from (2.14), (2.17) and the equality in (2.15) we have

$$\begin{aligned} c &> \left(\int_0^{a_k} f_k^2(r)\mathrm{d}r\right)^{1/2}\sqrt{a_k} - \int_0^{a_k} f_k(r)\mathrm{d}r \\ &\geq (v_k^2\epsilon_k a_k + w_k^2(1-\epsilon_k)a_k)^{1/2}\sqrt{a_k} - m_k\, a_k \\ &= (m_k^2 a_k^2 + \epsilon_k(1-\epsilon_k)a_k^2(v_k - w_k)^2)^{1/2} - m_k a_k, \end{aligned}$$

which directly implies

$$\epsilon_k(1-\epsilon_k)a_k^2(v_k - w_k)^2 < c^2 + 2cm_k\, a_k.$$

Finally, as $v_k \neq w_k$, this last inequality, (2.15) and (2.16) give

$$1 - \epsilon_k < \frac{c^2 + 2cm_k a_k}{\epsilon_k a_k^2 (v_k - w_k)^2} = \frac{c^2 + 2cm_k a_k}{a_k^2(m_k - w_k)(v_k - w_k)} < \frac{4(c^2 + 2cm_k a_k)}{m_k^2 a_k^2}$$

which concludes the proof. □

Lemma 2.6. *If* (H_2) *holds then there cannot exist a sequence* $(x_k)_k \subset \Omega^1(x_0, x_1)$ *satisfying both*

$$(J(x_k))_k \textit{ is bounded from above} \tag{2.18}$$

and

$$\|\dot{x}_k\| \to +\infty \quad \textit{as } k \to +\infty, \tag{2.19}$$

where functional J *is defined by* (2.3) *and* (2.4).

Proof. Assume by contradiction the existence of a sequence $(x_k)_k \subset \Omega^1(x_0, x_1)$ such that both (2.18) and (2.19) hold.

Firstly, let us notice that for each $x \in \Omega^1(x_0, x_1)$ we have

$$J(x) \geq \|\dot{x}\|^2 + \Sigma(x, \Delta_t)\left(\int_0^1 \frac{\mathrm{d}s}{\beta(x)}\right)^{-1} \tag{2.20}$$

with

$$\Sigma(x, \Delta_t) := \int_0^1 \frac{\langle\delta(x), \dot{x}\rangle^2}{\beta(x)}\mathrm{d}s \int_0^1 \frac{\mathrm{d}s}{\beta(x)} - \left(|\Delta_t| + \int_0^1 \frac{|\langle\delta(x), \dot{x}\rangle|}{\beta(x)}\mathrm{d}s\right)^2.$$

Hence, by (2.18) and (2.19) it has to be the case that

$$\Sigma(x_k, \Delta_t) < 0 \qquad \text{for } k \text{ large enough.} \tag{2.21}$$

On the other hand, from the Cauchy–Schwarz inequality we have

$$\int_0^1 \frac{\langle \delta(x), \dot{x}\rangle^2}{\beta(x)} \mathrm{d}s \geq \left(\int_0^1 \frac{|\langle \delta(x), \dot{x}\rangle|}{\beta(x)} \mathrm{d}s\right)^2 \left(\int_0^1 \frac{\mathrm{d}s}{\beta(x)}\right)^{-1}, \tag{2.22}$$

which replaced into (2.20) yields

$$J(x) \geq \|\dot{x}\|^2 - \left(\Delta_t^2 + 2|\Delta_t| \int_0^1 \frac{|\langle \delta(x), \dot{x}\rangle|}{\beta(x)} \mathrm{d}s\right) \left(\int_0^1 \frac{\mathrm{d}s}{\beta(x)}\right)^{-1}.$$

Therefore, if we define

$$\bar{J}(x) := \|\dot{x}\|^2 - \left(\Delta_t^2 + 2|\Delta_t| \int_0^1 \frac{|\langle \delta(x), \dot{x}\rangle|}{\beta(x)} \mathrm{d}s\right) \left(\int_0^1 \frac{\mathrm{d}s}{\beta(x)}\right)^{-1}, \tag{2.23}$$

necessarily by (2.18) it follows that

$$(\bar{J}(x_k))_k \text{ is bounded from above.} \tag{2.24}$$

In order to get a contradiction, first we need to rewrite the expression for $\bar{J}(x_k)$ in a “simpler” way. By (2.19), we can obviously assume $\|\dot{x}_k\| > 0$ for all $k \in \mathbb{N}$; so, we can define

$$\theta_k := \Delta_t^2 \left(\int_0^1 \frac{\|\dot{x}_k\|^2}{\beta(x_k)} \mathrm{d}s\right)^{-1},$$

$$P_k := \left(\int_0^1 \frac{|\langle \delta(x_k), \dot{x}_k\rangle|}{\beta(x_k)} \mathrm{d}s\right) \left(\int_0^1 \frac{\|\dot{x}_k\|^2}{\beta(x_k)} \mathrm{d}s\right)^{-1}.$$

By (2.19) and Lemma 2.4 we have

$$\lim_{k\to+\infty} \theta_k = 0; \tag{2.25}$$

therefore, we can also assume $0 \leq \theta_k < 1$ for all $k \in \mathbb{N}$ and define

$$\kappa_k := \frac{2|\Delta_t|}{1-\theta_k},$$

which clearly satisfies

$$\lim_{k\to+\infty} \kappa_k = 2|\Delta_t|. \tag{2.26}$$

Summarizing, if we replace these definitions into (2.23), we obtain

$$\bar{J}(x_k) = (1-\theta_k)(1-\kappa_k P_k)\|\dot{x}_k\|^2. \tag{2.27}$$

Now, write

$$a_k := \int_0^1 \frac{\mathrm{d}s}{\beta(x_k(s))}, \qquad m_k := \frac{1}{a_k} \int_0^1 \frac{|\langle \delta(x_k(s)), \dot{x}_k(s)\rangle|}{\beta(x_k(s))} \mathrm{d}s, \tag{2.28}$$

so that

$$P_k = \frac{m_k}{\|\dot{x}_k\|^2}. \tag{2.29}$$

Obviously, if (up to subsequences) we have

$$\lim_{k\to+\infty} P_k = 0, \tag{2.30}$$

then (2.19), (2.25), (2.26) and (2.27) yield a contradiction to (2.24). Hence, we can assume

$$\lim_{k\to+\infty} m_k = +\infty, \qquad \lim_{k\to+\infty} m_k a_k = +\infty, \tag{2.31}$$

(otherwise, either (2.19) or Lemma 2.4 implies (2.30)), and thus, a constant $A > 0$ exists such that

$$\|\dot{x}_k\|^2 \leq A m_k \qquad \text{for all } k \in \mathbb{N}. \tag{2.32}$$

Define $c := |\Delta_t|$ (which must be strictly positive by (2.21) and (2.22)) and

$$f_k(r) := |\langle \delta(x_k(s(r))), \frac{\mathrm{d}x_k}{\mathrm{d}s}(s(r))\rangle| \not\equiv 0 \qquad \text{with } \mathrm{d}r = \frac{\mathrm{d}s}{\beta(x_k(s))}.$$

Then, condition (2.21) reduces to inequality (2.14). Therefore, we can apply Lemma 2.5 and deduce

$$\epsilon_k > 1 - \frac{8c}{m_k a_k} - \frac{4c^2}{m_k^2 a_k^2}, \tag{2.33}$$

where ϵ_k is such that

$$|D_k| = \epsilon_k\, a_k \qquad \text{with } D_k := \left\{ r \in [0, a_k] : |\langle \delta(x_k(s(r))), \frac{\mathrm{d}x_k}{\mathrm{d}s}(s(r))\rangle| \geq \frac{m_k}{2} \right\}.$$

Or, equivalently,

$$\int_{\tilde{D}_k} \frac{\mathrm{d}s}{\beta(x_k(s))} = \epsilon_k a_k \tag{2.34}$$

with

$$\tilde{D}_k := \left\{ s \in I : |\langle \delta(x_k(s)), \dot{x}_k(s)\rangle| \geq \frac{m_k}{2} \right\}. \tag{2.35}$$

Therefore, from (2.2) and (2.35) we have on $\tilde{D}_k$:

$$\begin{aligned}(\sqrt{\lambda}\, \mathrm{length}(x_k(s)) + 1)|\dot{x}_k(s)| &\geq (\sqrt{\lambda} d(x_k(s), x_0) + 1)|\dot{x}_k(s)| \\ &\geq |\delta(x_k(s))||\dot{x}_k(s)| \geq |\langle \delta(x_k(s)), \dot{x}_k(s)\rangle| \geq \frac{m_k}{2},\end{aligned} \tag{2.36}$$

$\mathrm{length}(x_k(s))$ being the length of the portion of curve $x_k([0, s])$. In particular, from (2.36) and (2.32), it follows that

$$m_k \int_{\tilde{D}_k} \frac{\mathrm{d}s}{(\sqrt{\lambda}\ \mathrm{length}\ (x_k(s)) + 1)^2} \leq \frac{4}{m_k} \int_{\tilde{D}_k} |\dot{x}_k(s)|^2 \mathrm{d}s \leq 4A \quad \text{for all } k. \tag{2.37}$$

But notice that the Cauchy–Schwarz inequality and (2.32) imply

$$\mathrm{length}(x_k(s)) \leq \sqrt{A m_k s} \qquad \text{for all } s \in I;$$

whence

$$m_k \int_0^1 \frac{\mathrm{d}s}{(\sqrt{\lambda}\,\mathrm{length}(x_k(s))+1)^2} \geq m_k \int_0^1 \frac{\mathrm{d}s}{(\sqrt{\lambda A m_k s}+1)^2}$$
$$= \frac{2}{\lambda A}\left(\lg(\sqrt{\lambda A m_k}+1)+\frac{1}{\sqrt{\lambda A m_k}+1}-1\right),$$

which goes to infinity by (2.31). Thus, by (2.37) we must have

$$\lim_{k\to+\infty} m_k \int_{\tilde{D}_k^c} \frac{\mathrm{d}s}{(\sqrt{\lambda}\,\mathrm{length}(x_k(s))+1)^2} = +\infty, \tag{2.38}$$

where $\tilde{D}_k^c = I \setminus \tilde{D}_k$.

On the other hand, (2.28), (2.34) and (2.1) imply

$$(1-\epsilon_k)a_k = \int_{\tilde{D}_k^c} \frac{\mathrm{d}s}{\beta(x_k(s))} \geq \frac{d_k}{m_k} \tag{2.39}$$

with

$$d_k := m_k \int_{\tilde{D}_k^c} \frac{\mathrm{d}s}{(\sqrt{\lambda}\,\mathrm{length}(x_k(s))+1)^2}.$$

Therefore, from (2.39) and (2.33)we obtain:

$$d_k \leq (1-\epsilon_k)m_k a_k < 8c + \frac{4c^2}{m_k a_k}$$

in contradiction with (2.38) (recall (2.31)). □

Proof of Theorem 1.2. By Proposition 2.1 and previous related discussion, it is enough to prove that J has a minimum point. Hence, we just have to check that J satisfies the hypotheses of Theorem 2.3 in the complete Riemannian manifold $\Omega^1(x_0, x_1)$.

To this end, firstly we claim that J must be bounded from below. In fact, if it does not hold, a sequence $(x_k)_k$ exists such that $J(x_k) \to -\infty$. Then, necessarily (2.19) must hold (otherwise, $(\|x_k\|_\infty)_k$ has a bounded subsequence, and thus, $(J(x_k))_k$ also has), and Lemma 2.6 yields the contradiction.

Now, in order to prove that J satisfies the Palais–Smale condition, let $(x_k)_k \subset \Omega^1(x_0, x_1)$ and $M > 0$ be such that

$$\sup_{k\in\mathbb{N}} J(x_k) \leq M \quad \text{and} \quad \lim_{k\to+\infty} J'(x_k) = 0. \tag{2.40}$$

Clearly, Lemma 2.6 and (2.40) imply that $(\|\dot{x}_k\|)_k$ is bounded; whence, it follows that

$$\sup\{d(x_k(s), x_0) : s \in I, k \in \mathbb{N}\} < +\infty.$$

So, not only $(x_k)_k$ is bounded in $H^1(I, \mathbb{R}^N)$ but, by Proposition 2.1 and (2.4), also $(t_k)_k$ is bounded in $H^1(I, \mathbb{R})$ with $t_k = \Psi(x_k)$. Hence, there exist $x \in H^1(I, \mathbb{R}^N)$, $t \in H^1(I, \mathbb{R})$ such that, up to subsequences, it is $x_k \rightharpoonup x$ weakly in $H^1(I, \mathbb{R}^N)$ and $t_k \rightharpoonup t$ weakly in $H^1(I, \mathbb{R})$ (and also uniformly in I). As $\mathcal{M}_0$ is complete then $x \in \Omega^1(x_0, x_1)$ while $t \in W(t_0, t_1)$, so $\tau_k t_k - t \in H_0^1$ and by [5, Lemma 2.1] there exist two sequences $(\xi_k)_k, (\nu_k)_k \subset H^1(I, \mathbb{R}^N)$ such that

$$\xi_k \in T_{x_k}\Omega^1(x_0, x_1), \qquad x_k - x = \xi_k + \nu_k \qquad \text{for all } k \in \mathbb{N},$$
$$\xi_k \rightharpoonup 0 \text{ weakly} \qquad \text{and} \qquad \nu_k \to 0 \text{ strongly in } H^1(I, \mathbb{R}^N).$$

By Remark 2.2 we have

$$f'(z_k)[(\xi_k, -\tau_k)] = J'(x_k)[\xi_k] = o(1) \qquad \text{with } z_k = (x_k, t_k);$$

hence, reasoning as in the proof of [12, Lemma 3.4.1], we have $\xi_k \to 0$ strongly in $H^1(I, \mathbb{R}^N)$ and this implies $x_k \to x$ strongly in $\Omega^1(x_0, x_1)$.

Hence, J has a minimum point. □

Finally, we study the optimal/critical character of the hypotheses in Theorem 1.2. Observe that the accuracy of the assumption on β is ensured by the family of static spacetimes given in [3, Section 7]. So, in the following example we focus on hypothesis (1.5). Concretely, we construct a (non-static) stationary spacetime with coefficient β constantly equal to 1 and vector field δ superlinear (but as close as we want to linear) such that the corresponding functional J is unbounded from below on curves with certain fixed extreme points. Whence, we have shown that Theorem 1.2 essentially exhausts the variational technique for the study of geodesic connectedness in the (standard) stationary case. In particular, this example is a promising candidate for being non-geodesically connected.

Example 2.7. Consider the stationary spacetime $\mathcal{M} = \mathbb{R}^2 \times \mathbb{R}$ endowed with metric

$$\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle + 2\delta(x)\mathrm{d}x_2\mathrm{d}t - \mathrm{d}t^2$$

where $\langle \cdot, \cdot \rangle = \phi(x_2)\mathrm{d}x_1^2 + \mathrm{d}x_2^2$, $x = (x_1, x_2)$ are the natural coordinates of plane $\mathbb{R}^2$ and functions ϕ, respectively δ, satisfy

- $\phi(x_2)$ decreases as $x_2^{-2\alpha}$ when $x_2 \to +\infty$, for a fixed $\alpha > 0$;
- if $x_1 \le -1$ then $\delta(x_1, x_2) \equiv \delta(x_2)$ with $\delta(x_2)$ growing as $x_2^{1+\epsilon}$ when $x_2 \to +\infty$, and if $x_1 \ge 1$ then $\delta(x_1, x_2) \equiv \delta(x_2)$ with $\delta(x_2)$ decreasing as $-x_2^{1+\epsilon}$ when $x_2 \to +\infty$, for a fixed $\epsilon > 0$. For simplicity, let us assume that we have exactly $\delta(x_2) = x_2^{1+\epsilon}$ (respectively $\delta(x_2) = -x_2^{1+\epsilon}$) for x_2 large enough.

The idea is to choose a suitable sequence of diverging curves $(y_k)_k$ with fixed extremes such that $\langle \delta(y_k), \dot{y}_k \rangle$ remains essentially constant along the curve. Then, the superlinear character of δ will imply that $J(y_k) \to -\infty$.

Thus, consider the sequence of piecewise smooth curves $y_k : I \to \mathbb{R}^2$ defined in the following way:

$$y_k(s) = \begin{cases} \left(1, k^{\frac{1}{2+\epsilon}}\left(s + \frac{1}{k}\right)^{\frac{1}{2+\epsilon}}\right) & \text{if } s \in \left[0, \frac{1}{2} - \rho_k\right[\\ \left(-\frac{1}{\rho_k}\left(s - \frac{1}{2}\right), k^{\frac{1}{2+\epsilon}}\left(\frac{1}{2} - \rho_k + \frac{1}{k}\right)^{\frac{1}{2+\epsilon}}\right) & \text{if } s \in \left[\frac{1}{2} - \rho_k, \frac{1}{2} + \rho_k\right], \\ (-1, (-ks + k + 1)^{\frac{1}{2+\epsilon}}) & \text{if } s \in \left]\frac{1}{2} + \rho_k, 1\right], \end{cases}$$

with $\rho_k = k^{\frac{-\alpha}{2+\epsilon}}$. A direct computation shows

$$\int_0^1 \langle \dot{y}_k, \dot{y}_k \rangle \mathrm{d}s \le C(\epsilon)k, \qquad \int_0^1 \langle \delta(y_k), \dot{y}_k \rangle^2 \mathrm{d}s = \frac{1 - 2\rho_k}{(2+\epsilon)^2}k^2$$

$$\int_0^1 \langle \delta(y_k), \dot{y}_k \rangle \mathrm{d}s = -\frac{1 - 2\rho_k}{2+\epsilon}k,$$

where $C(\epsilon)$ does not depend on k. Summarizing, we can write

$$\begin{aligned} J(y_k) &\le \int_0^1 \langle \dot{y}_k, \dot{y}_k \rangle + \int_0^1 \langle \delta(y_k), \dot{y}_k \rangle^2 \\ &\quad - \left(\int_0^1 \langle \delta(y_k), \dot{y}_k \rangle \right)^2 + 2\Delta_t \int_0^1 \langle \delta(y_k), \dot{y}_k \rangle \\ &\le \left(C(\epsilon) - \frac{2\Delta_t (1 - 2\rho_k)}{2+\epsilon} \right) k + \frac{2\rho_k - 4\rho_k^2}{(2+\epsilon)^2} k^2. \end{aligned}$$

Therefore, if we take $\alpha > 2(2+\epsilon)$ and choose $\Delta_t = t_1 - t_0$ such that

$$\Delta_t > \frac{(2+\epsilon)C(\epsilon)}{2(1-2\rho_2)},$$

with $\rho_2 = 2^{-\frac{\alpha}{2+\epsilon}} \ge \rho_k$ for all $k \ge 2$, we obtain $J(y_k) \to -\infty$ as $k \to +\infty$.

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